

# Boundary Degeneracy of Topological Order

Juven Wang<sup>1,2</sup> and Xiao-Gang Wen<sup>2,1,3</sup>

<sup>1</sup>*Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

<sup>2</sup>*Perimeter Institute for Theoretical Physics, Waterloo, ON, N2L 2Y5, Canada*

<sup>3</sup>*Institute for Advanced Study, Tsinghua University, Beijing, 100084, P. R. China*

We introduce the notion of boundary degeneracy of topologically ordered states on a compact orientable spatial manifold with boundaries, and emphasize that it provides richer information than the bulk degeneracy. Beyond the bulk-edge correspondence, we find the ground state degeneracy of fully gapped edge states depends on boundary gapping conditions. We develop a quantitative description of different types of boundary gapping conditions by viewing them as different ways of non-fractionalized particle condensation on the boundary. Via Chern-Simons theory, this allows us to derive the ground state degeneracy formula in terms of boundary gapping conditions, which reveals more than the fusion algebra of fractionalized quasiparticles. We apply our results to Toric code and Levin-Wen string-net models. By measuring the boundary degeneracy on a cylinder, we predict  $Z_k$  gauge theory and  $U(1)_k \times U(1)_k$  non-chiral fractional quantum hall state at even integer  $k$  can be experimentally distinguished. Our work refines definitions of symmetry protected topological order and intrinsic topological order.

PACS numbers:

**Introduction**— Quantum many-body systems exhibit surprising new phenomena where topological order and the resulting fractionalization are among the central themes[1, 2]. One way to define topological order is through its ground state degeneracy(GSD) on two spatial dimensional(2D) higher genus closed Riemann surface, which is encoded by the fusion rules of fractionalized quasiparticles and the genus number[3]. However, on a 2D compact manifold with boundaries(Fig. 1), there can be gapless boundary edge states. For non-chiral topological order states, where the numbers of left and right moving modes equal, we show there are rules that edge states can be fully gapped out. It is the motivation of this paper to understand GSD for this type of systems where all boundary edge excitations are gapped. In the following, we name this degeneracy as boundary GSD, to distinguish it from bulk GSD of a gapped phase on a closed manifold without boundary. To understand the property of boundary GSD is both of theoretical interests and of application purpose where lattice model in topological quantum computation such as Toric code[4] can be put on space with boundaries[5, 6].

In this paper, we demonstrate boundary GSD is not simply a factorization of the degeneracies of all boundaries. Not only the fusion rules of fractionalized quasiparticles(anyons) and the manifold topology, but also boundary gapping conditions are the necessary data. Boundary GSD reveals richer information than bulk GSD. Moreover, gluing edge states of a compact manifold with boundaries to form a closed manifold, enables us to obtain bulk GSD from boundary GSD. We first introduce physical notions characterizing this boundary GSD and then rigorously derive its general formula by Chern-Simons theory[1]. For concrete physical pictures, we realize specific cases of our result by  $Z_2$  Toric code and string-net model[7]. Interestingly, by measuring bound-

ary GSD on a cylinder, our result predicts distinction between  $Z_k$  gauge theory( $Z_k$  Toric code) and  $U(1)_k \times U(1)_k$  non-chiral fractional quantum hall state at even integer  $k$ , despite the two phases cannot be distinguished by bulk GSD. By the same idea, we refine definitions of *symmetric protected topological order*(SPT) and *intrinsic topological order*. Our prediction can be tested experimentally.

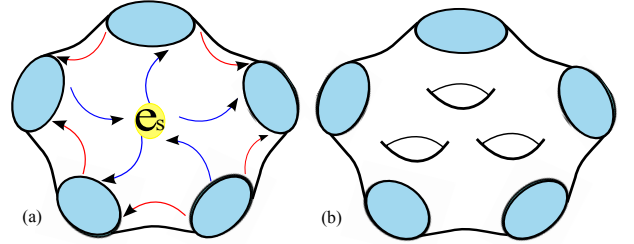


FIG. 1: (a) Illustration of *fusion rules* and *total neutrality*, where anyons are transported from one boundary to another(red arrows), or when they fuse into physical excitations(blue arrows) (b) A higher genus compact surface with boundaries(punctures).

**Physical Notions**— Topological order on a compact spatial manifold with boundaries have  $N$  branches of gapless edge states[1]. Suppose the manifold has total  $\eta$  boundaries, we label each boundary as  $\partial_\alpha$ , with  $1 \leq \alpha \leq \eta$ . Let us focus on the case the manifold is homeomorphic to a sphere with  $\eta$  punctures(Fig. 1(a)), we will comment on cases with genus or handles(Fig. 1(b)) later. If electrons condense on the boundary due to edge states scattering, it introduces mass gap to the edge states. A set of electrons can condense on the same boundary if they do not have relative quantum fluctuation phases with each other, thus all *condensed electrons* are stabilized in the classical sense. It requires *condensed electrons*

have relative zero Aharonov-Bohm (charge-flux) phase, we call these electrons are *null* and *mutual null*[8]. Since electrons have discrete elementary charge unit, we label them as a dimension- $N(\dim-N)$  lattice  $\Gamma_e$ , and label *condensed electrons* as discrete lattice vectors  $\ell^{\partial\alpha}(\in \Gamma_e)$  for boundary  $\partial_\alpha$ . We define a *complete set* of *condensed electrons*, labeled as lattice  $\Gamma^{\partial\alpha}$ , to include all electrons which are null and mutual null to each other. Notably there are different complete sets of condensed electrons, representing different kinds of boundary gapping conditions. Assigning a *complete set* of *condensed (non-fractionalized bosonic) electrons* to a boundary corresponds to assigning a type of boundary gapping condition. The number of types of *complete sets* equals to the number of types of boundary conditions. In principle each boundary can assign its own boundary condition independently, this assignment is not determined from the bulk information. Below we focus on the non-chiral orders, assuming all branches of edge states can be fully gapped out.

Remarkably there exists a set of *compatible anyons* do not produce flux effect to *condensed electron* charge, so their Aharonov-Bohm phases are zero. In other words, *compatible anyons* are mutual null to any elements in the *complete set* of *condensed electrons*. For boundary  $\partial_\alpha$ , We label *compatible anyons* as discrete lattice vectors  $\ell_{qp}^{\partial\alpha}$  and find all such anyons to form a *complete set* labeled as  $\Gamma_{qp}^{\partial\alpha}$ . Note that  $\Gamma^{\partial\alpha} \subset \Gamma_{qp}^{\partial\alpha}$ .  $\Gamma^{\partial\alpha}$  and  $\Gamma_{qp}^{\partial\alpha}$  have the same dimension. If *compatible anyons* can transport between different boundaries of the compact manifold, they must follow *total neutrality* - the net transport of *compatible anyons* between boundaries must be balanced by the fusion of physical particles in the system(Fig. 1(a)), so  $\sum_\alpha \ell_{qp}^{\partial\alpha} \in \Gamma_e$ . Transporting anyons from boundaries to boundaries in a fractionalized manner(i.e. not in integral electron units), changes topological sectors(i.e. ground states) of the system. Given data:  $\Gamma_e, \Gamma^{\partial\alpha}, \Gamma_{qp}^{\partial\alpha}$ , we thus derive generally GSD counts the number(Num) of elements in a quotient group:

$$\text{GSD} = \text{Num} \left[ \frac{\{(\ell_{qp}^{\partial 1}, \dots, \ell_{qp}^{\partial \eta}) \mid \forall \ell_{qp}^{\partial \alpha} \in \Gamma_{qp}^{\partial \alpha}, \sum_\alpha \ell_{qp}^{\partial \alpha} \in \Gamma_e\}}{\{(\ell^{\partial 1}, \dots, \ell^{\partial \eta}) \mid \forall \ell^{\partial \alpha} \in \Gamma^{\partial \alpha}\}} \right] \quad (1)$$

**Boundary degeneracy of Abelian topological order**— To demonstrate our above notions, let us take Abelian topological order as an example, which is believed to be fully classified by  $K$  matrix Abelian Chern-Simons theory[9]. For a system lives on a 2D compact manifold  $\mathcal{M}$  with 1D boundaries  $\partial\mathcal{M}$ , edge states of each closed boundary(homeomorphic to  $S^1$ ) are described by multiplet chiral bosons[1], with bulk and boundary actions:

$$S_{\text{bulk}} = \frac{K_{IJ}}{4\pi} \int_{\mathcal{M}} dt d^2x \epsilon^{\mu\nu\rho} a_\mu^I \partial_\nu a_\rho^J \quad (2)$$

$$S_\partial = \frac{1}{4\pi} \int_{\partial\mathcal{M}} dt dx K_{IJ} \partial_t \Phi_I \partial_x \Phi_J + V_{IJ} \partial_x \Phi_I \partial_x \Phi_J + \int_{\partial\mathcal{M}} dt dx \sum_a g_a \cos(\ell_{a,I} \cdot \Phi_I) \quad (3)$$

$K_{IJ}$  and  $V_{IJ}$  are symmetric integer  $N \times N$  matrices,  $a_\mu^I$  is the 1-form emergent gauge field's  $I$ -th component in the multiplet.  $\cos(\ell_{a,I} \cdot \Phi_I)$  is derived from local Hermitian gapping term  $\psi + \psi^\dagger = e^{i\ell_{a,I} \cdot \Phi_I} + e^{-i\ell_{a,I} \cdot \Phi_I}$ , where  $\ell_a$  has  $N$ -component with integer coefficients.

*Canonical quantization of  $K$  matrix Abelian Chern-Simons theory edge states*— In order to understand the energy spectrum or GSD of the edge theory, we study the ‘quantum’ theory, by canonical quantizing the boson field  $\Phi_I$ . Since  $\Phi_I$  is the compact phase of a fermion field, its bosonization has zero mode  $\phi_{0I}$  and winding momentum  $P_{\phi_J}$ , in addition to non-zero modes[10]:

$$\Phi_I(x) = \phi_{0I} + K_{IJ}^{-1} P_{\phi_J} \frac{2\pi}{L} x + i \sum_{n \neq 0} \frac{1}{n} \alpha_{I,n} e^{-inx \frac{2\pi}{L}} \quad (4)$$

The periodic boundary size is  $L$ . The conjugate momentum field of  $\Phi_I(x)$  is  $\Pi_I(x) = \frac{\delta L}{\delta(\partial_t \Phi_I)} = \frac{1}{4\pi} K_{IJ} \partial_x \Phi_J$ . With conjugation relation for zero modes:  $[\phi_{0I}, P_{\phi_J}] = i\delta_{IJ}$ , and generalized Kac-Moody algebra for non-zero modes:  $[\alpha_{I,n}, \alpha_{J,m}] = nK_{IJ}^{-1} \delta_{n,-m}$ . We thus have canonical quantized fields:  $[\Phi_I(x_1), \Pi_J(x_2)] = \frac{1}{2} i\delta_{IJ} \delta(x_1 - x_2)$ .

*Gapping Rules, Hamiltonian, and Hilbert Space*[11]— Let us determine the properties of  $\ell_a$  as *condensed electrons* and the set of allowed *boundary gapping lattice*  $\Gamma^\partial = \{\ell_a\}$ ,  $a$  labels the  $a$ -th vector in  $\Gamma^\partial$ . Without any symmetry constraint, then any gapping term is allowed if it is[11]: (1) Null and mutual null[8]:  $\forall \ell_a, \ell_b \in \Gamma^\partial$ ,  $\ell_{a,I} K_{IJ}^{-1} \ell_{b,J} = 0$ , This implies self statistics and mutual statistics are bosonic, and the excitation is local. Localized fields are not eliminated by self or mutual quantum fluctuations, so the condensation survives in the classical sense. (2) ‘Physical’ excitation:  $\ell_a \in \Gamma_e = \{\sum_J c_J K_{IJ} \mid c_J \in \mathbb{Z}\}$ , is an excitation of electron degree of freedom since it lives on the ‘physical’ boundary. (3) Completeness of  $\Gamma^\partial$ , defined by:  $\forall \ell_c \in \Gamma_e$ , if  $\ell_c K^{-1} \ell_c = 0$  and  $\ell_c K^{-1} \ell_a = 0$  with  $\forall \ell_a \in \Gamma^\partial$ , then  $\ell_c \in \Gamma^\partial$  must be true. It turns out rules (1)(2)(3) are not guaranteed to fully gap out edge states, we add an extra rule (4): The system is non-chiral. The signature of  $K$  ( $\equiv$  Num of positive eigenvalues – Num of negative eigenvalues) must be zero, i.e.  $N$  is even. Additionally, the dimension of  $\Gamma^\partial$  must be  $N/2$  and  $\sqrt{|\det K|} \in \mathbb{N}$ . Physically, rule (4) excludes violating examples such as odd rank( $\equiv$ rk)  $K$  matrix with central charge  $c - \bar{c} \neq 0$ , which universally has gapless chiral states. For instance, the dim-1 boundary gapping lattice:  $\{n(A, B, C) \mid n \in \mathbb{Z}\}$  of  $K_{3 \times 3} = \text{diag}\{1, 1, -1\}$ , with  $A^2 + B^2 - C^2 = 0$ , satisfies (1)(2)(3), but cannot fully gap out edge states.

Since any linear combinations of  $\ell_a \in \Gamma_e$  still satisfy (1)(2)(3)(4), we can regard  $\Gamma^\partial$  as an *infinite discrete lattice group*. As we solve exactly the number of types of boundary gapping conditions  $\mathcal{N}_g^\partial$ , at  $\text{rk}(K) = 2$ ,  $\mathcal{N}_g^\partial = 2$ . However, for  $\text{rk}(K) \geq 4$ ,  $\mathcal{N}_g^\partial = \infty$ , which is rather surprising. For example, as  $K_{4 \times 4} = \text{diag}\{1, 1, -1, -1\}$ , one can find dim-2 boundary gapping lattices,  $\{n(A, B, C, 0), m(0, C, B, A) \mid n, m \in \mathbb{Z}\}$ , where different sets of  $A^2 + B^2 - C^2 = 0$  give different lattices. There are *infinite* solutions of them[11].

To determine the mass gap of boundary states and show the gap is finite in the large system size limit  $L \rightarrow \infty$ , we will take large  $g$  coupling limit of Hamiltonian:  $-g_a \int_0^L dx \cos(\ell_{a,I} \cdot \Phi_I) \rightarrow \frac{1}{2} g_a (\ell_{a,I} \cdot \Phi_I)^2 L$ . Solve the quadratic Hamiltonian exactly, we find the spectrum  $E(n) = \sqrt{\Delta^2 + (\frac{n\pi}{L})^2 \delta_1^2 + (\frac{n\pi}{L})^2 \delta_2^2}$ , generically indicating edge states has finite gap  $\Delta$  independent of boundary size  $L$ . To count boundary GSD of this gapped system, we need to include the full cosine term for the lowest energy states - zero and winding modes:

$$\cos(\ell_{a,I} \cdot \Phi_I) \rightarrow \cos(\ell_{a,I} \cdot (\phi_{0I} + K_{IJ}^{-1} P_{\phi,J} \frac{2\pi}{L} x)) \quad (5)$$

The kinetic term  $H_{kin} = \frac{(2\pi)^2}{4\pi L} V_{IJ} K_{IJ}^{-1} K_{IJ}^{-1} P_{\phi,I} P_{\phi,J}$  has  $O(1/L)$  so can be neglected in  $L \rightarrow \infty$ , the cosine potential Eq. (5) dominates, use  $[\ell_{a,I} \phi_{0I}, \ell_{a,I'} K_{IJ}^{-1} P_{\phi,J}] = \ell_{a,I} K_{IJ}^{-1} \ell_{a,I'} \delta_{IJ} = 0$  by rule (1), we can safely expand the cosine term, then integrate  $L$ :

$$g_a \int_0^L dx \text{ Eq. (5)} = g_a L \cos(\ell_{a,I} \cdot \phi_{0I}) \delta_{(\ell_{a,I} \cdot K_{IJ}^{-1} P_{\phi,J}, 0)} \quad (6)$$

$\phi_0$  is periodic, so  $P_\phi$  forms a discrete lattice. By rule (2), so  $\cos(\ell_{a,I} \cdot \phi_{0I})$  are hopping terms along *condensed electron* vector  $\ell_{a,I}$  in sublattice of  $\Gamma^e$  in  $P_\phi$  lattice.  $P_\phi^{qp}$  represents *compatible anyons*  $\ell_{qp}$  which is mutual null to *condensed electrons*  $\ell$  by  $\ell K^{-1} P_\phi^{qp} = \ell K^{-1} \ell_{qp} = 0$ . By rule (1),  $\ell_{qp}$  parallels along some  $\ell$  vector. However,  $\ell_{qp}$  lives on quasiparticle lattice, i.e. unit integer lattice of  $P_\phi$  lattice. So  $\ell_{qp}$  is parametrized by  $\frac{1}{|\text{gcd}(\ell_a)|} \ell_{a,J}$ , with greatest common divisor  $|\text{gcd}(\ell_a)| \equiv \text{gcd}(|\ell_{a,1}|, |\ell_{a,2}|, \dots, |\ell_{a,N}|)$ .

For boundary  $\partial_\alpha$ , a *complete set of condensed electrons* forms *boundary gapping lattice*  $\Gamma^{\partial_\alpha} = \{\sum_a I_a^{\partial_\alpha} \ell_{a,I}^\partial \mid I_a^{\partial_\alpha} \in \mathbb{Z}\}$ . A *complete set of compatible anyons* forms the Hilbert space of winding modes on  $P_\phi$  lattice:

$$\Gamma_{qp}^{\partial_\alpha} = \{\ell_{qp,I}^{\partial_\alpha}\} = \{\sum_a j_a^{\partial_\alpha} \frac{\ell_{a,I}^{\partial_\alpha}}{|\text{gcd}(\ell_a^{\partial_\alpha})|} \mid j_a^{\partial_\alpha} \in \mathbb{Z}\} \quad (7)$$

or simply *anyon hopping lattice*. Anyon fusion rules and *total neutrality* essentially imply physical charge excitations can fuse into multiple anyon charges. The rules constraint the direct sum[12] of *anyon hopping lattice*  $\Gamma_{qp}^{\partial_\alpha}$

over all  $\eta$  boundaries must be on the  $\Gamma_e$  lattice,

$$L_{qp} \cap e \equiv \{ \bigoplus_{\alpha=1}^{\eta} \sum_a j_a^{\partial_\alpha} \frac{\ell_{a,I}^{\partial_\alpha}}{|\text{gcd}(\ell_a^{\partial_\alpha})|} \mid \forall j_a^{\partial_\alpha} \in \mathbb{Z}, \exists c_J \in \mathbb{Z}, \\ \ni \sum_{\alpha,a} j_a^{\partial_\alpha} \frac{\ell_{a,I}^{\partial_\alpha}}{|\text{gcd}(\ell_a^{\partial_\alpha})|} = \sum_J c_J K_{IJ} \} \quad (8)$$

GSD counts topological sectors distinct by fractionalized anyons transport between boundaries. With  $\bigoplus_\alpha \Gamma^{\partial_\alpha}$  is a normal subgroup of  $L_{qp} \cap e$ , and given input data:  $K$  and  $\Gamma^{\partial_\alpha}$  (which determines  $\Gamma_{qp}^{\partial_\alpha}$ ), we derive GSD is the number of elements in a *quotient finite Abelian group*:

$$\text{GSD} = \text{Num} \left[ \frac{L_{qp} \cap e}{\bigoplus_\alpha \Gamma^{\partial_\alpha}} \right] \quad (9)$$

analog to Eq. (1). Interestingly Eq. (9) works for both closed manifolds or compact manifolds with boundaries[13]. By gluing boundaries of compact manifold, enlarging  $K$  matrix of glued edges states to  $K_{2N \times 2N}$ , creating  $N$  scattering channels to fully gap out edge states. For a genus  $g$  Riemann surface with  $\eta'$  punctures (Fig. 1(b)), we start with a number of  $g$  cylinders drilled with extra punctures[11], use Eq. (9), remove glued boundaries part which contributes a factor  $\leq |\det K|^g$ , and redefine particle hopping lattices  $L_{qp} \cap e$  and  $\bigoplus_{\alpha'} \Gamma^{\partial_{\alpha'}}$  only for unglued boundaries ( $1 \leq \alpha' \leq \eta'$ ),

$$\text{GSD} \leq |\det K|^g \cdot \text{Num} \left[ \frac{L_{qp} \cap e}{\bigoplus_{\alpha'=1}^{\eta'} \Gamma^{\partial_{\alpha'}}} \right] \quad (10)$$

For genus  $g$  Riemann surface ( $\eta' = 0$ ), Eq. (10) becomes  $\text{GSD} \leq |\det K|^g$ . The inequalities are due to different choices of gapping conditions for glued boundaries[13].

Apply our algorithm to generic  $K_{2 \times 2}$  matrix case[11], to fully gap out edge states require  $\det K = -k^2$  with integer  $k$ . Take a cylinder (a sphere with 2 punctures) as an example, Eq. (9) shows  $\text{GSD} = \sqrt{|\det K|} = k$  when boundary conditions on two edges are the same  $\Gamma^{\partial_1} = \Gamma^{\partial_2}$ . However,  $\text{GSD} \leq k$  when  $\Gamma^{\partial_1} \neq \Gamma^{\partial_2}$ . Specifically, take  $K_{Z_k} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$  and  $K_{\text{diag},k} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$ , we obtain  $K_{Z_k}$  has  $\text{GSD} = 1$ , while  $K_{\text{diag},k}$  has  $\text{GSD} = 1$  for odd  $k$  but  $\text{GSD} = 2$  for even  $k$ . See Table I.

		$K_{Z_k}$	$K_{\text{diag},k}$
Boundary	$\Gamma^{\partial_1} \neq \Gamma^{\partial_2}$	1	1 ( $k \in \text{odd}$ ) or 2 ( $k \in \text{even}$ )
	$\Gamma^{\partial_1} = \Gamma^{\partial_2}$	$k$	$k$
Bulk GSD		$k^2$	$k^2$

TABLE I: Boundary GSD and bulk GSD for  $K_{Z_k}$  and  $K_{\text{diag},k}$

This shows a new surprise, we predict a distinction between two types of order  $K_{Z_k}$  ( $Z_k$  gauge theory) and  $K_{\text{diag},k}$  ( $U(1)_k \times U(1)_k$  non-chiral fractional quantum hall state) at even integer  $k$  by simply measuring their boundary GSD on a cylinder.

**Mutual Chern-Simons,  $Z_k$  gauge theory, Toric code and String-net model**— We now take  $K_{Z_k}$  example to demonstrate our understanding of two types of GSD on a cylinder in physical pictures.

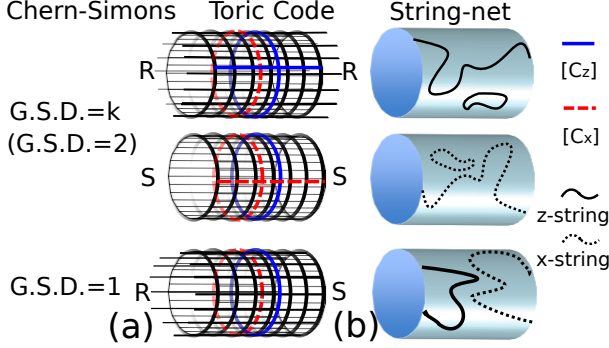


FIG. 2: (a) Same boundary conditions on two ends of a cylinder allow a pair of cycles  $[c_x], [c_z]$  of a qubit, thus  $\text{GSD} = 2$ . Different boundary conditions do not, thus  $\text{GSD} = 1$ . (b) Same boundary conditions allow  $z$ - or  $x$ -strings connect two boundaries. Different boundary conditions do not.

When  $k = 2$ , it realizes  $Z_2$  Toric code[14] with Hamiltonian[4]  $H_0 = -\sum_v A_v - \sum_p B_p$ . There are two types of boundaries[5] on a cylinder(Fig. 2(a)):  $x$  (R: rough) boundary where  $z$ -string charge  $e$  condenses and  $z$  (S: smooth) boundary where  $x$ -string charge  $m$  condenses. We can determine GSD by counting the degree of freedom of the code subspace: Num of qubits – Num of independent stabilizers. For  $\Gamma^{\partial_1} = \Gamma^{\partial_2}$ , we have the same number of qubits and stabilizers, with one extra constraint  $\prod_{all} B_p = 1$  for two  $x$ -boundaries ( $\prod_{all} A_v = 1$  for two  $z$ -boundaries). This leaves 1 free qubit, thus  $\text{GSD} = 2$ . For  $\Gamma^{\partial_1} \neq \Gamma^{\partial_2}$ , still the same number of qubits and stabilizers, but has no extra constraint. This leaves no free qubits, thus  $\text{GSD} = 1$ . We can also count independent logical operators(Fig. 2(a)) in homology class, with string-net(Fig. 2(b)) picture in mind - there are two cycles  $[c_{x_1}], [c_{z_1}]$  winding around the compact direction of a cylinder. If both are  $x$ -boundaries, we only have  $z$ -string connects two edges: cycle  $[c_{z_2}]$ . If both are  $z$ -boundaries, we only have  $x$ -string (dual string) connects two edges: cycle  $[c_{x_2}]$ . Cycles of either case define the algebra  $\sigma_x, \sigma_y$  of a qubit, so  $\text{GSD} = 2$ . For both  $x$ -boundaries ( $z$ -boundaries), one ground state has even number of strings (dual strings), the other ground state has odd number of strings (dual strings), connecting two edges. If boundaries are different, no cycle is allowed in the non-compact direction, no string and no dual string can connect two edges, so  $\text{GSD} = 1$ . Generally, for a level  $k$  doubled model,  $\text{GSD} = k^{\dim[H_1(\mathcal{M}; \mathbb{Z}_2)]} = k^{b_1(\mathcal{M})}$ ,  $k$  to the power of the 1st Betti number[13, 15].

**Definitions of Topological Order**— Now let us ask a fundamental question: what is topological order? We realize the original definition of degenerated ground states

on a higher genus Riemann surface can be transplanted to degenerated ground states on a cylinder with two boundaries. For a non-chiral fully-gapped-boundary state, we define: *the state is intrinsic topological order, if it has degenerated ground states (at least for certain boundary gapping conditions) on a cylinder*. The at least statement is due to  $\text{GSD} \leq \sqrt{|\det K|}$ , only the same boundary conditions on two sides gives  $\text{GSD} = \sqrt{|\det K|}$ . Similarly, *the state is trivial order or SPT, if it has a unique ground state on a cylinder for any boundary gapping condition*.

**Conclusion**— In summary, by  $K_{N \times N}$  matrix Abelian Chern-Simon theory, we derive boundary fully gapping conditions for Abelian topological order states, its low energy Hamiltonian and Hilbert space, a general GSD formula, and the number of types of boundary gapping conditions  $\mathcal{N}_g^\partial$ . The fully gapped boundaries requires  $N = \text{even}$  and non-chiral(*doubled*) Chern-Simons theory, reflecting gapped boundary *Quantum Doubled* model of Toric code and string-net[6, 16]. Our  $\mathcal{N}_g^\partial$  formula proves the two boundary types conjecture of Toric code[5, 15, 16],  $Z_k$  gauge theory, and a more general  $K_{2 \times 2}$  Chern-Simon theory. We show gapping edge states to count boundary GSD is related to bulk GSD. However, we find there are more types of boundary GSD (than a unique bulk GSD) depending on types of boundary gapping conditions. A remarkable example is rank-2  $K_{Z_k}$  and  $K_{\text{diag}, k}$  with  $k = \text{even}$ [11], where we predict experimentally  $\text{GSD} = k$  or 1 for  $K_{Z_k}$  and  $\text{GSD} = k$  or 2 for  $K_{\text{diag}, k}$  on a cylinder, though GSD of both phases on a closed genus  $g$  surface are indistinguishable( $= k^{2g}$ ). This example is especially surprising because both phases  $K_{Z_k}$  and  $K_{\text{diag}, k}$  have the same  $(\mathbb{Z}_k)^2$  fusion algebra. This means fusion algebra alone cannot determine boundary GSD[11]. In the category language[6], the model of unitary fusion category  $\mathcal{C}$  shows that (in Table II) there can be many different  $\mathcal{C}, \mathcal{D}$  types realizing the same monoidal center  $Z(\mathcal{C}) = Z(\mathcal{D})$ .

Physics	Category
Bulk excitation (anyons)	objects in unitary modular category $Z(\mathcal{C})$ (monoidal center of $\mathcal{C}$ )
Boundary type	the set of equivalent classes $\{\mathcal{C}, \mathcal{D}, \dots\}$ of unitary fusion category

TABLE II: Dictionary between physics and category

Finally, our definitions of *topological orders* not only deepen understanding of topological GSD, but also ease the experimental platform with only cylinder topology instead of higher genus surfaces. To open future research avenues, it will be interesting to realize boundary GSD and  $\mathcal{N}_g^\partial$  without using  $K$  matrix, which is restricted only to Abelian topological order, also to study them on higher dimensional space. Physical notions which we introduced should still hold universally. Other than *fusion rule* and *total neutrality*, whether *braiding rule* can enter



into GSD formula[17]? Additionally, engineer boundaries to line-like defects may synthesize projective non-Abelian statistics[18]. All these shall inspire generalizing Eq. (9) to boundary GSD of non-Abelian topological order.

It will be illuminating to have more predictions based on our theory, as well as experimental realizations of boundary types. One approach is flux insertion through a cylinder, where the adiabatic flux change induces anyon transport from one boundary to the other by  $\Delta\Phi_B/(h/e) = \Delta P_\phi$ [11]. It will be interesting to see how the same type of boundary conditions allow this effect (unit flux changes topological sectors, with total sectors as  $\text{GSD} = \sqrt{|\det K|}$ ), while different type of boundary conditions restrain this effect ‘dynamically’ ( $\text{GSD} < \sqrt{|\det K|}$ ). To detect this dynamical effect can guide experiments to distinguish boundary types.

We thank Maissam Barkeshli, William Witczak and Lucy Zhang for comments on the paper. This work is supported by NSF Grant No. DMR-1005541, NSFC 11074140, and NSFC 11274192. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research.

---

[1] X. -G. Wen, Adv. Phys. **44** 405 (1995)

[2] F. Wilczek, (ed.), *Fractional statistics and anyon super-*

*conductivity* (World Scientific, 1990).

[3] X. G. Wen, Phys. Rev. B **40**, 7387 (1989).

[4] A. Y. Kitaev, Annals Phys. **303**, 2 (2003)

[5] S. B. Bravyi, A. Y. Kitaev, arXiv: quant-ph/9811052.

[6] A. Kitaev, L. Kong, Commun. Math. Phys. **313**, 351 (2012)

[7] M. A. Levin and X. -G. Wen, Phys. Rev. B **71**, 045110 (2005)

[8] F. D. M. Haldane, Phys. Rev. Lett. **74**, 2090 (1995).

[9] X. G. Wen and A. Zee, Phys. Rev. B **46**, 2290 (1992).

[10] X. G. Wen, Phys. Rev. B **41**, 12838 (1990).

[11] Explicit results in the appendix of this paper.

[12] For Abelian group, finite direct product is finite direct sum, so  $\bigoplus_\alpha \Gamma^{\partial_\alpha} = \prod_\alpha \Gamma^{\partial_\alpha}$ , with  $\alpha$  a finite number of punctures. Direct sum is indeed the coproduct in the category of abelian groups.

[13] J. Wang, X.-G. Wen, in preparation.

[14] S. -P. Kou, M. Levin and X. -G. Wen, Phys. Rev. B **78**, 155134 (2008)

[15] M. F. Freedman, D. A. Meyer, arXiv:quant-ph/9810055,

[16] S. Beigi, P.W. Shor, D. Whalen, Commun. Math. Phys. **306**, 663 (2011)

[17] Peculiarly simply apply Eq. (9) to surfaces without genus (Fig. 1(a)) and with genus (Fig. 1(b)) cases gives the same GSD, if both have the same boundary number and boundary conditions. However by gluing techniques, we learn from Eq. (10) there can be a factor contribution ( $|\det K|^g$ ) from the genus topology. So the nontrivial winding in genus may modify Eq. (9), where braiding rules may involve.

[18] M. Barkeshli, C. -M. Jian and X. -L. Qi, arXiv:1208.4834 [cond-mat.str-el].

## APPENDIX

In Appendix, we explain detailed physical meanings of boundary gapping rules in Sec. A, and demonstrate our algorithm and GSD formula Eq. (9) for a generic rank-2  $K$  matrix in Sec. B. We also give an example why fusion algebra alone does not provide enough information to determine boundary GSD from the bulk-edge correspondence viewpoint. In Sec. C, we comment more about the number of boundary types  $\mathcal{N}_g^\partial$ . In Sec. D, we use gluing techniques to derive Eq. (10) for a compact manifold with genus. Lastly, we explain how flux insertion experiment may help to test boundary types in Sec. E.

### A. BOUNDARY GAPPING RULES

We argue boundary gapping rules are:

(i) Bosonic:  $\forall \ell_a \in \Gamma^\partial$ ,  $\ell_{a,I} K_{IJ}^{-1} \ell_{a,J} \in \text{even integer}$ , which means mutual statistics has multiple  $2\pi$  phase.

(ii) Local:  $\forall \ell_a, \ell_b \in \Gamma^\partial$ ,  $\ell_{a,I} K_{IJ}^{-1} \ell_{b,J} \in \text{integer}$ . Winding one excitation around another yields multiple  $2\pi$  phase.

(iii) Localizing field at classical value without being

eliminated by self or mutual quantum fluctuation:  $\forall \ell_a, \ell_b \in \Gamma^\partial$ ,  $\ell_{a,I} K_{IJ}^{-1} \ell_{b,J} = 0$ , so the condensation survives in the classical sense.

(iv)  $\ell_a$  must be excitations of electron degree of freedom since it lives on the ‘physical’ boundary, so  $\ell_a \in \Gamma_e$  electron lattice, where  $\Gamma_e = \{\sum_J c_J K_{IJ} \mid c_J \in \mathbb{Z}\}$ . This rule imposes integer charge  $q_I K_{IJ}^{-1} \ell_{a,J}$  in the bulk, and integer charge  $Q_I = \int_0^L \frac{1}{2\pi} \partial_x \Phi_I dx = K_{IJ}^{-1} P_{\phi_J} = K_{IJ}^{-1} \ell_{a,J}$  for each branch on the boundary. Here  $q_I$  is the charge vector coupling to an external field  $A_\mu$  of gauge or global symmetry, by adding  $A_\mu q_I J_I^\mu$  to the  $S_{\text{bulk}}$ , which causes  $q_I A_\mu \partial_x \Phi_I$  in the  $S_\partial$ .

(v) Completeness: we define  $\Gamma^\partial$  is a complete set by:  $\forall \ell_c \in \Gamma_e$ , if  $\ell_c K^{-1} \ell_c = 0$  and  $\ell_c K^{-1} \ell_a = 0$  with  $\forall \ell_a \in \Gamma^\partial$ , then  $\ell_c \in \Gamma^\partial$  must be true, otherwise  $\Gamma^\partial$  is not complete. The completeness rule is due to no symmetry protection forbids any possible local gapping terms.

We can summarize all above rules of  $\Gamma^\partial$  as (1) Null and mutual null:  $\forall \ell^a, \ell^b \in \Gamma^\partial$ ,  $\ell_{a,I} K_{IJ}^{-1} \ell_{b,J} = 0$ , (2) ‘Physical’ excitation:  $\ell_a \in \Gamma_e$ , (3) Completeness of  $\Gamma^\partial$ . (1)(2)(3) are not guaranteed to fully gap out edge states, we add an extra rule: (4)  $N \in \text{even}$ . Dimension of  $\Gamma^\partial$  must be  $N/2$ , which requires  $\sqrt{|\det K|} \in \mathbb{N}$ .

## B. EXACT RESULTS OF $K_{2 \times 2}$ CHERN-SIMONS THEORY

Here we work through a rank-2  $K$  matrix Chern-Simons theory example, to demonstrate our generic algorithm in the main text. We derive its low energy Hamiltonian, Hilbert space, boundary GSD formula, and the number of types of boundary gapping conditions  $\mathcal{N}_g^\partial$ . For  $K_{2 \times 2} = \begin{pmatrix} k_1 & k_3 \\ k_3 & k_2 \end{pmatrix} \equiv \begin{pmatrix} k_1 & k_3 \\ k_3 & (k_3^2 - p^2)/k_1 \end{pmatrix}$ , in order to fully gap out edge states, we find that  $\det K = -p^2$ , (a) the edge states need to be non-chiral ( $K_{2 \times 2}$  has equal number of positive and negative eigenvalues, so  $\det K < 0$  and (b)  $|\det K|$  needs to be an integer  $p$  square. Two independent sets of allowed gapping lattices  $\Gamma^\partial = \{n\ell_{a,I} | n \in \mathbb{Z}\}$ ,  $\Gamma^{\partial'} = \{n'\ell'_{a,I} | n' \in \mathbb{Z}\}$  satisfy gapping rules (1)(2)(3)(4), so  $\mathcal{N}_g^\partial = 2$  at  $\text{rk}(K) = 2$ , with

$$n\ell_{a,I} = n(\ell_{a,1}, \ell_{a,2}) = \frac{np}{|\text{gcd}(k_1, k_3 + p)|} (k_1, k_3 + p) \quad (11)$$

$$n'\ell'_{a,I} = n'(\ell'_{a,1}, \ell'_{a,2}) = \frac{n'p}{|\text{gcd}(k_3 + p, k_2)|} (k_3 + p, k_2) \quad (12)$$

$|\text{gcd}(k, l)|$  stands for the greatest common divisor in  $|k|, |l|$ , its absolute value. If  $k$  (or  $l$ ) is zero, we define  $|\text{gcd}(k, l)|$  is the other value  $|l|$  (or  $|k|$ ).  $n, n' \in \mathbb{Z}$  are allowed if no other symmetry constrains its values.

Now take specific topology, a disk (a sphere with 1 puncture) and a cylinder (a sphere with 2 punctures) as examples of manifolds with boundary, for  $K_{2 \times 2}$  their Hilbert space of edge states are [1]:

$$\mathcal{H}_{\text{disk}} = \mathcal{H}_{KM}^{1,2} \otimes \mathcal{H}_{P_{\phi_1}} \otimes \mathcal{H}_{P_{\phi_2}} \quad (13)$$

(if  $K_{2 \times 2}$  is diagonal, then  $\mathcal{H}_{KM}^{1,2} = \mathcal{H}_{KM}^1 \otimes \mathcal{H}_{KM}^2$ ).

$$\mathcal{H}_{\text{cylinder}} = \bigoplus_{j_A} \bigoplus_{j_B} (\mathcal{H}_{\text{disk}}^{\text{top}} \otimes \mathcal{H}_{\text{disk}}^{\text{bottom}})^{(j_A, j_B)} \quad (14)$$

$$= \mathcal{H}_{\text{disk}}^{\text{top}} \otimes \mathcal{H}_{\text{disk}}^{\text{bottom}} \otimes \mathcal{H}_{gl} \quad (15)$$

$$= \mathcal{H}_{KM}^{\text{top}, 1, 2} \otimes \mathcal{H}_{KM}^{\text{bottom}, 3, 4} \otimes \mathcal{H}_{P_{\phi_1}^{\text{top}}, P_{\phi_3}^{\text{bottom}}} \otimes \mathcal{H}_{P_{\phi_2}^{\text{top}}, P_{\phi_4}^{\text{bottom}}}$$

$\mathcal{H}_{KM}$  stands for Hilbert space of nonzero Fourier modes part with Kac-Moody algebra. We label zero and winding modes' Hilbert space by winding mode  $P_\phi$ , which can be regarded as a discrete lattice because of  $\Phi_I(x)$  periodicity. Because the bulk cylinder provides channels connecting edges states of two boundaries, so fractionalized quasiparticles (here Abelian anyons) can be transported from one edge to the other.  $\mathcal{H}_{gl}$  contains fractional sectors  $|j_A, j_B\rangle$ , the 1st branch  $j_A$  runs between the top( $P_{\phi_1}^{\text{top}}$ ) and the bottom( $P_{\phi_3}^{\text{bottom}}$ ), the 2nd branch  $j_B$  runs between the top( $P_{\phi_2}^{\text{top}}$ ) and the bottom( $P_{\phi_4}^{\text{bottom}}$ ).

Let us explicitly show that edge states of any boundary has a finite gap from ground states at large system size  $L$  and large coupling  $g$ . Without losing generality, take  $V = \begin{pmatrix} v_1 & v_2 \\ v_2 & v_1 \end{pmatrix}$  and a gapping term  $(\ell_{a,1}, \ell_{a,2}) =$

$\frac{p}{|\text{gcd}(k_1, k_3 + p)|} (k_1, k_3 + p) \in \Gamma^\partial$ , and diagonalize the Hamiltonian,

$$H \simeq \left( \int_0^L dx V_{IJ} \partial_x \Phi_I \partial_x \Phi_J \right) + \frac{1}{2} g (\ell_{a,I} \cdot \Phi_I)^2 L \quad (16)$$

we find eigenvalues:

$$E_{1,2}(n) = \sqrt{\Delta^2 + \left(\frac{n\pi}{Lp^2}\right)^2} \delta_1 \pm \left(\frac{n\pi}{Lp^2}\right) \delta_2 \quad (17)$$

where the finite mass gap is independent of  $L$ :

$$\Delta = \frac{\sqrt{2\pi g (k_1(k_1 - k_2)v_1 + 2(k_3 + p)(k_3v_1 - k_1v_2))}}{|\text{gcd}(k_1, k_3 + p)|},$$

$\delta_1 = (k_1 - k_2)^2 v_1^2 + 4(k_3v_1 - k_1v_2)(k_3v_1 - k_2v_2)$  and  $\delta_2 = v_1(k_1 + k_2) - v_2(2k_3)$ .

To count boundary GSD, for generic  $K_{2 \times 2}$  Abelian topological order on a disk, take  $\ell_a \in \Gamma^\partial$  in Eq. (11) without losing generality (same argument for  $\Gamma^{\partial'}$ ) [2], we have  $P_{\phi_1} = I_1 \ell_{a,1} + j_A \frac{\ell_{a,1}}{|\text{gcd}(\ell_a)|}$ ,  $P_{\phi_2} = I_1 \ell_{a,2} + j_A \frac{\ell_{a,2}}{|\text{gcd}(\ell_a)|}$ . The total anyon charge for each branch needs to conserve, but a single boundary of a disk has no other boundaries to locate transported anyons. This implies:  $j_A = 0$ , there is no different topological sector induced by transporting anyons, thus  $\text{GSD} = 1$ .

On the other hand, if the topology is replaced to a cylinder with the top  $\partial_1$  and the bottom  $\partial_2$  boundaries in Fig. 3, when the gapping terms from *boundary gapping lattice*  $\Gamma^\partial$  are chosen to be the same, the Hilbert space on  $P_\phi$  lattice is:

$$P_{\phi_1} = I_1 \ell_{a,1} + j_A \frac{\ell_{a,1}}{|\text{gcd}(\ell_a)|}, P_{\phi_2} = I_1 \ell_{a,2} + j_A \frac{\ell_{a,2}}{|\text{gcd}(\ell_a)|} \quad (18)$$

$$P_{\phi_3} = I_2 \ell_{a,1} + j_B \frac{\ell_{a,1}}{|\text{gcd}(\ell_a)|}, P_{\phi_4} = I_2 \ell_{a,2} + j_B \frac{\ell_{a,2}}{|\text{gcd}(\ell_a)|} \quad (19)$$

Anyon fusion rule and charge conservation for each branch constrains  $(P_{\phi_1} + P_{\phi_3}, P_{\phi_2} + P_{\phi_4}) \in \Gamma_e$  electron lattice. With  $|\text{gcd}(\ell_a)| = p$ , it implies  $j_A = -j_B \pmod{p}$ .  $0 \leq j_A \pmod{p} < p$  has  $p$  different topological sectors induced by different  $j_A$ . When  $\Delta j_A / p \in \mathbb{Z}$ , it transports electrons, so it brings back to the same sector. Count the number of distinct sectors, i.e. ground states, we find  $\text{GSD} = p$ .

If gapping terms on two boundaries of a cylinder are chosen to be different:  $\Gamma^\partial$  for  $\partial_1$ ,  $\Gamma^{\partial'}$  for  $\partial_2$ , we revise the second line of Eq. (19) to

$$P_{\phi_3} = I_2 \ell'_{a,1} + j'_B \frac{\ell'_{a,1}}{|\text{gcd}(\ell'_a)|}, P_{\phi_4} = I_2 \ell'_{a,2} + j'_B \frac{\ell'_{a,2}}{|\text{gcd}(\ell'_a)|} \quad (20)$$

Anyon fusion rules and conservation implies:  $(\frac{k_1}{|\text{gcd}(\ell_a)|} j_A + \frac{k_3 + p}{|\text{gcd}(\ell'_a)|} j'_B, \frac{k_3 + p}{|\text{gcd}(\ell_a)|} j_A + \frac{k_2}{|\text{gcd}(\ell'_a)|} j'_B) \in \Gamma_e$ . This constraint gives a surprise. For example,  $K_{Z_p} = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$ ,  $\text{GSD} = 1$ . However, when

$K_{\text{diag},p} = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}$ ,  $\text{GSD} = 1$  for  $p \in \text{odd}$ , but  $\text{GSD} = 2$  for  $p \in \text{even}$ . This provides a new approach that one can distinguish two types of orders  $K_{Z_p}$  and  $K_{\text{diag}}$  when  $p$  is even by measuring their boundary GSD.

We illustrate this result in an intuitive way in Fig. 3. When boundary types are the same on two sides of the cylinder, Fig. 3(a) is enough to explain  $\text{GSD} = p$ , where fractionalized anyons transport from the bottom to the top. For  $K_{Z_p}$  case, say  $\Gamma^{\partial_1} = \Gamma^{\partial_2} = \Gamma^\partial = \{n(p, 0)\}$ , qp1 with  $\ell_a = j_A(1, 0)$  for  $0 \leq j_A \leq p-1$ . For  $K_{\text{diag},p}$  case, say  $\Gamma^{\partial_1} = \Gamma^{\partial_2} = \Gamma^\partial = \{n(p, p)\}$ , qp1 with  $\ell_a = j_A(1, 1)$  for  $0 \leq j_A \leq p-1$ . This accounts all  $p$  sectors.

When boundary types are different, Fig. 3(a) is not allowed for fractionalized anyons transport. Fig. 3(b) is crucial to account the second ground state of  $K_{\text{diag},p}$  at  $p \in \text{even}$ . Let us take  $\Gamma^{\partial_1} = \Gamma^\partial = \{n(p, p)\}$  and  $\Gamma^{\partial_2} = \Gamma^{\partial'} = \{n(p, -p)\}$ , where  $e_s$  represents  $\ell = (p, 0)$ , while qp1 with  $\ell_a = (p/2, p/2)$  and qp2 with  $\ell_a = (p/2, -p/2)$  at  $p \in \text{even}$  are allowed fractionalized anyons (with integer unit of anyon charge). This process switches the ground state to a different sector, so  $\text{GSD} = 2$ . However, fractionalized anyons transport in Fig. 3(b) is not allowed for  $K_{Z_p}$  with different boundary types on two sides of the cylinder, which results in  $\text{GSD} = 1$ .

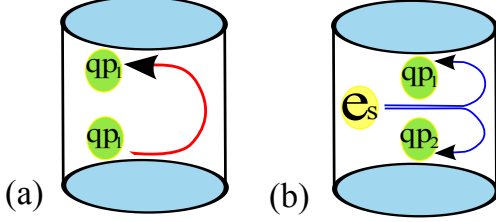


FIG. 3: (a) Anyon(qp1) is transported from the bottom to the top of the cylinder. (b) Physical excitations  $e_s$  split into a pair of anyons(qp1 to the bottom, qp2 to the top). Note that (b) is crucial to explain  $\text{GSD} = 2$  for  $K_{\text{diag},p}$  at  $p \in \text{even}$  with different boundary types on two sides of the cylinder.

The result remarks that only fusion algebra(both have doubled fusion algebra  $(\mathbb{Z}_p)^2$ ) is not sufficient enough to determine boundary GSD. Let us show this by  $K_{Z_2}$  and  $K_{\text{diag},2}$  examples from the bulk-edge correspondence viewpoint. Both orders have the doubled fusion algebra,  $(\mathbb{Z}_2)^2$ , with bulk excitation such as two anyon types  $\ell_a = (0, 1), \ell_b = (1, 0)$ . For  $K_{Z_2}$ , two types of boundary conditions  $\Gamma^\partial = \{n2(0, 1) \mid n \in \mathbb{Z}\}$  and  $\Gamma^\partial = \{n2(1, 0) \mid n \in \mathbb{Z}\}$  correspond to two of  $\ell_a, \ell_b$ , thus this is one to one correspondence. However, for  $K_{\text{diag},2}$ , two boundary conditions  $\Gamma^\partial = \{n2(1, 1) \mid n \in \mathbb{Z}\}$  and  $\Gamma^{\partial'} = \{n2(1, -1) \mid n \in \mathbb{Z}\}$  correspond to one bulk excitation  $\ell = (1(\text{mod}2), 1(\text{mod}2)) = (1(\text{mod}2), -1(\text{mod}2))$  in fusion algebra. It is two (on the boundary) to one (in the bulk) correspondence. Hence  $K_{\text{diag},2}$  shows an example where anyon fusion algebra from the *bulk* cannot distinguish different *boundary* types.

### C. NUMBER OF TYPES OF BOUNDARY GAPPING CONDITIONS

For the number of types of boundary gapping conditions  $\mathcal{N}_g^\partial$ , at  $\text{rk}(K) = 2$ , we showed  $\mathcal{N}_g^\partial = 2$ , which proves the conjecture[3] nicely.  $\text{rk}(K) \geq 4$ ,  $\mathcal{N}_g^\partial$  is *infinite*, is surprising. We may wonder whether extra rules are possible to restrict  $\mathcal{N}_g^\partial$ . Let us assume  $|\det K| = 1$ , the canonical form of this unimodular indefinite symmetric integral  $K_{N \times N}$  matrix exists[4]. For the odd matrix(where quadratic form has some odd integer coefficient, so with fermionic statistics), the canonical form is composed by  $N/2$  blocks of  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  along the diagonal blocks of  $K_{N \times N}$ . For the even matrix(where quadratic form has only even integer coefficient, so with only bosonic statistics), the canonical form is composed by blocks of  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and a set of all positive(or negative) coefficients  $E_8$  lattices,  $K_{E_8}$ ,

$$K_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (21)$$

along the diagonal blocks of  $K_{N \times N}$ . A positive definite  $K_{E_8}$  with eight chiral bosons cannot be gapped out. Thus, in order to have non-chiral states, the even matrix canonical form must be composed by  $N/2$  blocks of  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Now it is useful to rethink about the number of boundary types  $\mathcal{N}_g^\partial$  in this canonical form when  $|\det K| = 1$ . We had claimed under rule (1)(2)(3)(4) when  $\text{rk}(K) \geq 4$ ,  $\mathcal{N}_g^\partial = \infty$ . If we add an extra gapping rule (5): Each gapping term is formed by only two branches of chiral-antichiral edge states scattering. There are no three or more multi-channel scatterings. With (5), the boundary gapping lattices  $\Gamma^\partial$  are formed by a set of  $\ell$  terms of two branches of edge states with opposite chirality. This modifies our result to  $\mathcal{N}_g^\partial = (N/2)!2^{N/2}$ . Its  $(N/2)!$  factor is due to each chiral state can find another state from antichiral ones to form a pair. The relative sign of their bosonized phase is  $\pm$ , gives an extra  $2^{N/2}$  factor. We comment that (5) seems to be artificial without deep reasoning, and the formula only works for  $|\det K| = 1$  with the canonical form. However, one can postulate some extra rule like (5) can restrict  $\mathcal{N}_g^\partial$ .

### D. SURGERY TO GLUE CYLINDERS TO FORM A GENUS $g$ RIEMANN SURFACE WITH PUNCTURES

Here we show how to glue the boundaries of punctured cylinders to form a genus  $g$  Riemann surface with punc-

tures, and determine its GSD of the topological order.

For a genus  $g$  Riemann surface with  $\eta'$  punctures (Fig. 1(b) and Fig. 4(c)), we start from Fig. 4(a), a number of  $g$  cylinders drilled with total puncture number  $\eta = \eta' + 2g + 2(g - 1)$ , where  $2g$  count two punctures on top and bottom for each cylinder ( $h_{i,T}$  and  $h_{i,B}$ ,  $1 \leq i \leq g$ ), drill an extra puncture on both the 1st ( $h_{1,L}$ ) and the last  $g$ th cylinder ( $h_{g-1,R}$ ), and drill two extra punctures ( $h_{j-1,R}$  and  $h_{j,L}$ ) for  $j$ -th cylinder for  $2 \leq j \leq g - 1$ . There are thus  $2(g - 1)$  extra punctures. Glue the boundaries of  $h_{j,L}$  and  $h_{j,R}$  together for  $1 \leq j \leq g - 1$ , and glue  $h_{i,T}$  and  $h_{i,B}$  together for  $1 \leq i \leq g$ , results in Fig. 4(c). Use Eq. (9), remove glued boundaries part ( $1 \leq \alpha \leq 2g + 2(g - 1)$ ) which contributes

a factor  $\leq |\det K|^g$ , and redefine particle hopping lattices  $L_{qp \cap e}$  and  $\bigoplus_{\alpha'} \Gamma^{\partial_{\alpha'}}$  only for unglued boundaries ( $1 \leq \alpha' \leq \eta'$ ), we derive

$$\text{GSD} \leq |\det K|^g \cdot \text{Num} \left[ \frac{L_{qp \cap e}}{\bigoplus_{\alpha'=1}^{\eta'} \Gamma^{\partial_{\alpha'}}} \right] \quad (22)$$

For a genus  $g$  Riemann surface ( $\eta' = 0$ ), this becomes  $\text{GSD} \leq |\det K|^g$ , where  $\text{rk}(K) = N$  for a closed manifold case is relaxed to any natural number  $N$ , which works for both odd and even number of branches. The inequalities here are due to different choices of gapping conditions for glued boundaries.

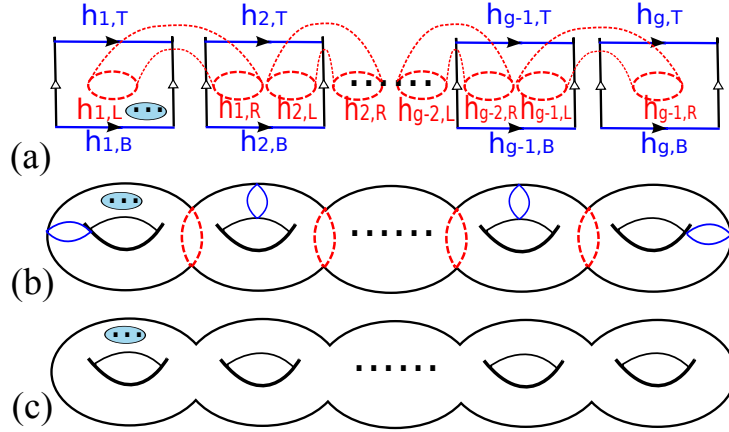


FIG. 4: Glue punctured cylinders to form a genus  $g$  Riemann surface with  $\eta'$  punctures. Start from (a), firstly identify left and right  $\triangleright$  arrows of each square to form number  $g$  of punctured cylinders. Then glue  $h_{j,L}$  and  $h_{j,R}$  (red dotted circles) together for  $1 \leq j \leq g - 1$ , and glue  $h_{i,T}$  and  $h_{i,B}$  (blue arrows) together for  $1 \leq i \leq g$ , which yields (b), equivalently as a genus  $g$  Riemann surface (c). The extra  $\eta'$  punctures are indicated here as a shaded blue puncture in the left most handle.

## E. FLUX INSERTION ARGUMENT AND EXPERIMENTAL TEST ON BOUNDARY TYPES

Consider an artificial-designed or external gauge field (such as electromagnetic field) coupled to topologically ordered states by a charge vector  $q_I$ . An adiabatic flux insertion  $\Delta\Phi_B$  inside the cylinder induces  $E_x$ , causing a perpendicular current  $J_y$  flows to the boundary, from the bulk term  $J_J^\mu = -q_I \frac{e}{2\pi} K_{IJ}^{-1} \frac{c}{\hbar} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$ , so  $q_I \Delta\Phi_B = -q_I \int dt \int \vec{E} \cdot d\vec{l} = -\frac{2\pi}{e} K_{IJ} \hbar \int J_{y,J} dt dx = -\frac{2\pi}{e} K_{IJ} \frac{\hbar}{e} Q_J$ .  $Q_J$  is the charge condensed on the edge of the cylinder. On the other hand, the edge state dynamics affects winding modes by  $Q_I = \int J_{\partial,I}^0 dx = -\int \frac{e}{2\pi} \partial_x \Phi_I dx = -e K_{IJ}^{-1} P_{\phi,J}$ , so

$$q_I \Delta\Phi_B / \left(\frac{\hbar}{e}\right) = \Delta P_{\phi,I} \quad (23)$$

For the same types of boundaries:  $\Gamma^{\partial_1} = \Gamma^{\partial_2}$ , there are  $\sqrt{|\det K|}$  sectors by  $\Delta P_\phi \pmod{\sqrt{|\det K|}}$ , the  $\sqrt{|\det K|}$  units of flux bring the state back to the original sector. For  $\Gamma^{\partial_1} \neq \Gamma^{\partial_2}$ , we had shown  $\text{GSD} < \sqrt{|\det K|}$  (such as  $\text{GSD} = 1$ ). This motivates an interesting question if one inserts flux into the cylinder, what dynamical effect, which repulses anyons transporting from one edge to the other, will be detected. The detection of this effect may guide experiments to distinguish different types of boundary gapping conditions.

[1] X. G. Wen, Phys. Rev. B **41**, 12838 (1990).

[2] Notice if we take  $|n| \geq 2$  or  $|n'| \geq 2$  larger-size hopping term for  $n\ell_{a,I} \in \Gamma^{\partial}$ ,  $n'\ell'_{a,I} \in \Gamma^{\partial'}$ , the lattice can break into more sublattices of different sectors: the calculation



on a disk is altered to  $\text{GSD} = |n|$ . However, generically no symmetry forbids  $n = n' = 1$  least-size hopping terms, which dominant cosine potential can minimize the energy.

[3] S. B. Bravyi, A. Y. Kitaev, arXiv: quant-ph/9811052.

[4] <http://mathoverflow.net/questions/97448>